STANFIELD, LINDA LOUISE. Some Banach Algebras of Analytic Functions. (1970) Directed by: Dr. Hughes B. Hoyle, III. pp. 31

The purpose of the paper is to investigate the ideal structure in two Banach Algebras: A, the space of all complex-valued functions which are continuous on the closed unit disc $\bar{U}$ in the complex plane and analytic on the open unit disc U, and H^∞, the space of all bounded analytic functions on U. Some basic concepts of Banach Algebras are developed; the existence of a one-to-one correspondence between the set of all maximal ideals of a Banach Algebra X and the set $\mathcal{M}(X)$ of all homomorphisms from X to $\mathbb{C}$ is shown; a topology with respect to which $\mathcal{M}(X)$ is a compact Hausdorff space is exhibited. The investigation of the ideal structure in A and H^∞ shows that every maximal ideal in A is the kernel of an evaluation on A; $\mathcal{M}(A)$ is homeomorphic to $\bar{U}$; each $z \in \bar{U}$ with $|z| = 1$ corresponds to infinitely many ideals in $\mathcal{M}(H^\infty)$. Finally, it is shown that the Silov boundary of H^∞ is a proper subset of those ideals in $\mathcal{M}(H^\infty)$ which correspond to points $z \in \bar{U}$ where $|z| = 1$.

SOME BANACH ALGEBRAS OF ANALYTIC FUNCTIONS

by

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A Thesis Submitted to
the Faculty of the Graduate School at
The University of North Carolina at Greensboro
in Partial Fulfillment
of the Requirements for the Degree
Master of Arts

Greensboro
August, 1970

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ACKNOWLEDGMENT

I would like to express my sincere gratitude to Dr. Nelson F. Page for his ingenuity and assistance in the preparation of this thesis.

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INTRODUCTION

The purpose of this paper is to investigate the ideal structure in two Banach Algebras: A, the space of all complex-valued functions which are continuous on the closed unit disc $\bar{U}$ in the complex plane and analytic on the open unit disc U, and H^{∞}, the space of all bounded analytic functions on U. In Chapter I, we develop some basic concepts of Banach Spaces and Banach Algebras. In Chapter II, we show the existence of a one-to-one correspondence between the set of all maximal ideals of a Banach Algebra X and the set $\mathcal{M}(X)$ of all homomorphisms from X to $\mathbb{C}$. In Chapter III, we show the existence of a topology with respect to which $\mathcal{M}(X)$ is a compact Hausdorff space. From this topology we show the existence of a norm-preserving homomorphism from X onto a subalgebra of continuous functions on a compact Hausdorff space.

Chapter IV is an investigation of the ideal structures in the two Banach Algebras, A and H^{∞}. We prove that every maximal ideal in A is the kernel of an evaluation map on A, where an evaluation map on A is a mapping ϕ_z with $\phi_z(f) = f(z)$ for each $f \in A$ and $z \in \bar{U}$. Equivalently, every multiplicative linear functional on A, i.e., every $L \in \mathcal{M}(A)$, is an evaluation. We also show that $\mathcal{M}(A)$ is homeomorphic to $\bar{U}$. We find the space of maximal ideals in H^{∞} to be much more complicated. Each $z \in \bar{U}$ with $|z| = 1$ corresponds to infinitely many ideals in $\mathcal{M}(H^{\infty})$.

Consideration of the Silov boundary for H^{∞} seems to be an enlightening way to further investigate $M(H^{\infty})$. A first guess might be that the Silov boundary of H^{∞} is the set of all those ideals corresponding to points $z \in \bar{U}$ where $|z| = 1$. This is shown not to be the case; the Silov boundary of H^{∞} is a proper subset of $M(H^{\infty}) - \{\phi_z : |z| < 1\}$.

CHAPTER I

BANACH SPACES AND BANACH ALGEBRAS

It is the purpose of this chapter to develop the concepts of Banach Spaces and Banach Algebras which will be useful in subsequent chapters.

Note: In this paper we will be considering only linear spaces over $\mathbb{C}$, the field of complex numbers.

Definition 1: A linear space (vector space) X is said to be a normed linear space if to each $f \in X$ there is associated a nonnegative real number $||f||$, called the norm of f, such that

i) $||f|| = 0$ iff $f = 0$, i.e., f is the additive identity for X,

ii) $||f + g|| \leq ||f|| + ||g||$ for all $f,g \in X$,

iii) $||zf|| = |z|\ ||f||$ for $f \in X$ and $z \in \mathbb{C}$.

Remark 1: By Definition 1 ii), the triangle inequality, $||f - g|| \leq ||f - h|| + ||h - g||$ holds. Combining i), ii), and iii) and taking $z = 0$ and $z = -1$, we have that every normed linear space is a metric space, the distance between f and g being $||f - g||$.

Definition 2: A Banach Space is a normed linear space which is complete in the metric defined by its norm.

Definition 3: An algebra is a linear space X in which an associative and distributive multiplication is defined, i.e.,

i) $f(gh) = (fg)h$,

$(f + g)h = fh + gh$,

$f(g + h) = fg + fh$ for $f,g,h \in X$,

and which is related to scalar multiplication so that

ii) $z(fg) = f(zg) = (zf)g$ for $f,g \in X$, z a scalar.

Note: In this paper we will be considering only commutative algebras with identity.

Definition 4: If an algebra X is a normed linear space under the norm $\|\ \|$ and satisfies the multiplicative inequality $\|fg\| \leq \|f\|\ \|g\|$, $f,g \in X$, then X is a normed algebra. If, in addition, X is a Banach Space and $\|\mathbb{1}\| = 1$, where $\mathbb{1}$ is the identity for X, then X is called a Banach Algebra.

Definition 5: Suppose J is a subspace of a linear space X, and associate with each $f \in X$ the coset $f + J = \{f + g : g \in J\}$. The set of all cosets of J is denoted by X/J and is called the quotient space of X with respect to J.

Note: It can be shown that X/J is a linear space.

Definition 6: A subset J of an algebra X is said to be an ideal if

i) J is a subspace of X,

ii) $fg \in J$ whenever $f \in X$, $g \in J$.

If $J \neq X$, J is a proper ideal. A maximal ideal is a proper ideal which is not contained in any other proper ideal.

Definition 7: An element f of an algebra X will be called invertible if f has an inverse in X, i.e., if there exists an element $f^{-1} \in X$ such that $f^{-1}f = ff^{-1} = 1$.

Theorem 1: Let X be an algebra and J a proper ideal in X. Then X/J is a field iff J is a maximal ideal.

Proof: Let $f \in X$, $f \notin J$, and $I = \{gf + h : g \in X, h \in J\}$. Then I is an ideal in X which properly contains J, since $f \in I$. If J is maximal, then $I = X$ and $gf + h = 1$ for some $g \in X$ and $h \in J$. Hence $(g + J)(f + J) = 1 + J$ which implies that every nonzero element of X/J is invertible. Therefore X/J is a field.

If J is not maximal, we can choose f as above so that $I \neq X$. Hence $1 \notin I$, and thus $f + J$ is not invertible in X/J. Therefore X/J is not a field.

Definition 8: If X is a normed linear space and J is a closed subspace of X we can define a norm for the quotient space X/J by $||f + J|| = \inf \{||f + g|| : g \in J\}$. We call this norm the quotient norm of X/J.

Theorem 2: The quotient norm of X/J has the following properties:

i) X/J is a normed linear space,

ii) If X is a Banach Space, so is X/J,

iii) If X is a Banach Algebra and J is a proper closed ideal then X/J is a Banach Algebra [7].

Definition 9: A mapping T of a linear space X into a linear space Y is called a linear transformation if $T(zf + g) = z\,T(f) + T(g)$ for all $f,g \in X$ and all $z \in \mathbf{C}$.

Definition 10: If T is a linear transformation from a normed linear space X into a normed linear space Y, we define its norm by

$$||T|| = \sup \left\{ \frac{||Tf||}{||f||} : f \in X,\ f \neq 0 \right\}.$$

If $||T|| < \infty$, then T is called a bounded linear transformation. Note that $||Tf|| \leq ||T||\ ||f||$ for each $f \in X$.

Theorem 3: Let T be a linear transformation from a normed linear space X into a normed linear space Y. Then

$$||T|| = \sup_{||f|| = 1} ||Tf|| = \sup_{||f|| \leq 1} ||Tf||.$$

Proof: Let $||T||_1 = \sup\limits_{||f|| = 1} ||Tf||$ and $||T||_2 = \sup\limits_{||f|| \leq 1} ||Tf||$.

Clearly $||T||_1 \leq ||T||_2$. Suppose $||T||_1 < ||T||_2$. Then there exists $f \in X$ such that $||f|| < 1$ and $||T||_1 < ||Tf||$. Let $g = \frac{1}{||f||} f$. Then we have $||T||_1 < ||Tf|| < ||Tg|| \leq ||T||_1$. This contradiction shows that $||T||_1 = ||T||_2$. Now suppose $||Tf|| \leq M\ ||f||$ for each $f \in X$. Then $M \geq ||T||_1$. Therefore $||T|| \geq ||T||_1$. Since $||Tf|| = ||T\ (\frac{1}{||f||}\ ||f||\ f)\ || = ||f||\ ||T\ (\frac{1}{||f||}\ f)\ || \leq ||f||\ ||T||_1$ we know $||T|| \leq ||T||_1$. Therefore $||T|| = ||T||_1 = ||T||_2$.

Theorem 4: If T is a linear transformation of a normed linear space X into a normed linear space Y, then these are equivalent:

i) T is bounded,

ii) T is continuous,

iii) T is continuous at 0,

iv) T is continuous at one point of X.

Proof: Suppose T is bounded. Then $||Tf_1 - Tf_2|| \le ||T||\ ||f_1 - f_2|| < \epsilon$ for all $f_1, f_2 \in X$ with $||f_1 - f_2|| < \epsilon/||T||$. Thus T is uniformly continuous and i) implies ii). It is trivial that ii) implies iii) and iii) implies iv). Suppose T is continuous at $f_0 \in X$. Then for each $\epsilon > 0$, we can find a $\delta > 0$ such that $||f - f_0|| < \delta$ implies $||T(f) - T(f_0)|| < \epsilon$, i.e., $||f|| < \delta$ implies $||T(f + f_0) - T(f_0)|| < \epsilon$. By linearity of T we have $||Tf|| < \epsilon$. Thus $||T|| \le \epsilon/\delta$ and iv) implies i).

Theorem 5: The space $B(X,Y)$ of all bounded linear transformations from a normed linear space X to a Banach Space Y is itself a Banach Space.

Proof: For T_1 and T_2 in $B(X,Y)$ we define $z\,T_1 + \xi\,T_2$ by $(z\,T_1 + \xi\,T_2)f = z\,T_1\,f + \xi\,T_2 f$. Clearly $z\,T_1 + \xi\,T_2$ is a linear transformation. Now $||z\,T_1|| = \sup\limits_{||f||=1} ||z\,T_1 f|| = |z| \sup\limits_{||f||=1} ||T_1 f||$

$= |z|\ ||T_1||$ and $||T_1 + T_2|| = \sup\limits_{||f||=1} ||T_1 f + T_2 f||$

$\le \sup\limits_{||f||=1} (||T_1 f|| + ||T_2 f||) \le ||T_1|| + ||T_2||$.

Thus any linear combination of bounded linear transformations is again a bounded linear transformation. If $||T_1|| = 0$, then

$||T_1 f|| \le ||T_1||\ ||f|| = 0$, and $T_1 f = 0$. Thus $||T_1|| = 0$ only for the transformation $T = 0$, which maps every $f \in X$ into the zero element of Y. Hence $||\ ||$ satisfies all requirements of a norm.

Let $\{T_n\}_{n=0}^{\infty}$ be a Cauchy sequence in $B(X,Y)$. Then for each $f \in X$ we have $||T_n f - T_m f|| \le ||T_n - T_m||\ ||f||$, and so $\{T_n f\}_{n=0}^{\infty}$ is a Cauchy sequence in Y and must converge to an element in Y. We will call this element Tf. By definition of Tf, $T(zf) = z\,Tf$ and $T(f + g) = Tf + Tg$.

Now, given $\epsilon > 0$ there is an N such that for all $m, n \ge N$ we have $||T_n - T_m|| < \epsilon$. Hence $||T_n|| < ||T_N|| + \epsilon$ for all $n \ge N$, and thus we have $||Tf|| = \lim_{n\to\infty} ||T_n f|| \le (||T_N|| + \epsilon)\ ||f||$. Hence T is bounded.

For each $f \in X$ we have $||T_n f - Tf|| =$
$\lim_{m\to\infty} ||T_n f - T_m f|| \le \overline{\lim_{m\to\infty}} ||T_n - T_m||\ ||f|| \le \epsilon\ ||f||$ for all $n \ge N$.
Thus for $n \ge N$ we have $||T_n - T|| = \sup_{||f|| = 1} ||(T_n - T)f|| \le \epsilon$.

Hence $T_n \to T$ and $B(X,Y)$ is complete.

Definition 11: A linear functional on a linear space X is a linear transformation from X to the space $\mathbb{C}$ of complex numbers.

Definition 12: If X and Y are algebras, a mapping ϕ of X into Y is said to be a homomorphism if ϕ is a linear transformation which also preserves multiplication, i.e., $\phi(f)\ \phi(g) = \phi(fg)$ for all $f,g \in X$. The kernel of ϕ, denoted $\ker\ \phi$, is the set of all $f \in X$ such that $\phi(f) = 0$.

<u>Remark 2</u>: It is trivial to verify that the kernel of a homomorphism is an ideal.

<u>Theorem 6</u>: If ϕ is a homomorphism from an algebra X to $\mathbb{C}$, then $||\phi|| = 1$.

<u>Proof</u>: Assume $||\phi|| > 1$. Then $|\phi(f)| > ||f||$ for some $f \in X$. Let $z = \phi(f)$ and let $g = f/z$. Then $||g|| < 1$ and $\phi(g) = 1$. Since $||g^n|| \leq ||g||^n$ and $||g|| < 1$, the sequence $\{s_n\}_1^\infty$ defined by $s_n = -g - g^2 - \cdot \cdot \cdot - g^n$ is a Cauchy sequence in X. Since X is complete, being a Banach Space, there exists $h \in X$ such that $\lim_{n\to\infty} ||h - s_n|| = 0$. Clearly $g + s_n = gs_{n-1}$, so $g + h = gh$. Hence $\phi(g) + \phi(h) = \phi(g)\,\phi(h)$ which contradicts $\phi(g) = 1$. Therefore $||\phi|| \leq 1$.

Now suppose $||\phi|| < 1$. Then $|\phi(1)| \leq ||\phi||\,||1|| < 1$. But $|\phi(1)| = |1| = 1$, a contradiction. Therefore $||\phi|| = 1$.

<u>Definition 13</u>: The <u>dual</u> of a normed space X, denoted X^*, is the space of bounded linear functionals on X.

<u>Remark 3</u>: Since $\mathbb{C}$ is complete, the dual X^* of any normed linear space X is a Banach Space by Theorem 5.

<u>Remark 4</u>: The reader is reminded of the following corollary to the Hahn-Banach Theorem: Let f be an element in a normed linear space X. Then there is a bounded linear functional L on X such that $L(f) = ||f||$ [6].

CHAPTER II

MAXIMAL IDEALS AND HOMOMORPHISMS

The purpose of this chapter is to show the existence of a one-to-one correspondence between the set of all maximal ideals of any Banach Algebra X and the set of all homomorphisms from X to $\mathbb{C}$.

Consider the following setting for Theorems 7 through 10: Let X be a Banach Algebra.

Theorem 7: If $||1 - f|| < 1$, $f \varepsilon X$, then f is invertible.

Proof: Let $g = 1 - f$. Then $||g|| < 1$ and $\sum_{k=0}^{\infty} g^k$ converges. Let $h = \sum_{k=0}^{\infty} g^k$, then $(1 - g)\, h = \lim_{n\to\infty} (1 - g)\, s_n = \lim_{n\to\infty} g^0 - g^{n+1} = 1$, where $s_n = \sum_{k=0}^{n} g^k$. Thus $1 - g$ is invertible and therefore f is invertible by definition of g.

Theorem 8: If $|z| > ||f||$, $f \varepsilon X$, then $f - z$ is invertible.

Proof: Since $||1 - (1/z\, f - 1)|| = ||1/z\, f|| \leq \frac{||f||}{|z|} < 1$, we know $1/z\, f - 1$ is invertible by Theorem 7. Clearly if $1/z\, f - 1$ is invertible, then $z(1/z\, f - 1)$ is invertible and since $f - z = z(1/z\, f - 1)$, the theorem is proved.

Theorem 9: The invertible elements of X form an open set.

Proof: Let f be an invertible element of X. Consider $S = \left\{ g \varepsilon X : ||f - g|| < \frac{1}{||f^{-1}||} \right\}$. For each $g \varepsilon S$ we have $||1 - f^{-1} g|| = ||f^{-1} f - f^{-1} g|| \leq ||f^{-1}||\ ||f - g|| < 1$. Therefore $f^{-1} g$ is invertible and $g = ff^{-1} g$ is invertible.

<u>Theorem 10</u>: Let V denote the set of invertible elements of X. The mapping $f \to f^{-1}$ is a continuous function on V.

<u>Proof</u>: Let $f \in V$ and choose $\varepsilon > 0$. Let

$\delta = \min\left(\frac{1}{2||f^{-1}||}, \frac{\varepsilon}{2||f^{-1}||^2}\right)$. Suppose $||f - g|| < \delta$.

Then $||f - g|| < \frac{1}{2||f^{-1}||}$,

$$||f^{-1}||\ ||f - g|| < 1/2 ,$$

$$||1 - f^{-1} g|| < 1/2 < 1, \text{ which}$$

implies $f^{-1} g$ is invertible and g is invertible. From the above inequalities we know that $1 - ||1 - f^{-1} g|| > 1/2$ and $||1 - f^{-1} g|| \le ||f^{-1}||\ ||f - g|| < \frac{\epsilon}{2||f^{-1}||}$. So

$\frac{||1 - f^{-1} g||}{1 - ||1 - f^{-1} g||} < \frac{\epsilon}{||f^{-1}||}$. For $||f - g|| < \delta$ we have

$$\frac{1}{f^{-1} g} = \sum_{k=0}^{\infty} (1 - f^{-1} g)^k,$$

$$\frac{1}{f^{-1} g} - 1 = \sum_{k=1}^{\infty} (1 - f^{-1} g)^k,$$

$$\frac{1 - f^{-1} g}{f^{-1} g} = \sum_{k=1}^{\infty} (1 - f^{-1} g)^k,$$

$$g^{-1} - f^{-1} = f^{-1} \sum_{k=1}^{\infty} (1 - f^{-1} g)^k,$$

$$||g^{-1} - f^{-1}|| \le ||f^{-1}|| \sum_{k=1}^{\infty} ||1 - f^{-1} g||^k$$

$$= ||f^{-1}|| \frac{||1 - f^{-1} g||}{1 - ||1 - f^{-1} g||} \le ||f^{-1}|| \frac{\epsilon}{||f^{-1}||} = \epsilon .$$

<u>Definition 14</u>: Let X be a Banach Algebra and $f \in X$. The <u>spectrum</u> of f, denoted $\sigma(f)$, is the name given to $\{z \in \mathbb{C}: f - z$ is not invertible$\}$.

<u>Remark 5</u>: From Theorem 8, if $z \in \sigma(f)$, then $|z| \leq ||f||$.

<u>Theorem 11</u>: Let X be a Banach Algebra and $f \in X$. The spectrum of f is nonempty and compact.

<u>Proof</u>: Let $f \in X$, $f \neq 0$ and suppose $\sigma(f)$ is empty. Let $L \in X^*$. Define F on $\mathbb{C}$ by: $F(z) = L(\frac{1}{f - z})$. Then

$$F'(z) = \lim_{w \to 0} \frac{F(z + w) - F(z)}{w} = \lim_{w \to 0} 1/w \left[L\left(\frac{1}{f - z - w}\right) - L\left(\frac{1}{f - z}\right)\right]$$

$$= \lim_{w \to 0} 1/w\ L\left(\frac{1}{f - z - w} - \frac{1}{f - z}\right) = \lim_{w \to 0} 1/w\ L\left(\frac{w}{(f - z - w)(f - z)}\right)$$

$$= \lim_{w \to 0} L\left(\frac{1}{(f - z - w)(f - z)}\right) = L\left(\lim_{w \to 0} \frac{1}{(f - z - w)(f - z)}\right)$$

$= L\left(\frac{1}{(f - z)^2}\right)$. Thus F is an entire function. Since

$\lim_{|z| \to \infty} F(z) = \lim_{|z| \to \infty} 1/z \left(L\left(\frac{1}{1/z\ f - 1}\right)\right) = 0$, there exists a positive number M such that $|z| > M$ implies $|F(z)| < 1$. Now we know F is bounded on the closed disc $\{z : |z| \leq M\}$ and $|F(z)| < 1$ outside the disc. Hence F is a bounded function. By Liouville's Theorem [3], F is constant. Suppose c, the constant value of F, is not zero. Let $\epsilon = |c|$. Since $\lim_{|z| \to \infty} F(z) = 0$, there exists $z \in \mathbb{C}$ such that $|F(z)| < \epsilon$, a contradiction. Thus $F(z) = 0$ for all z, hence $L(1/f) = 0$. We have shown that $L(1/f) = 0$ for each L in X^*. By Remark 3, this implies that $1/f = 0$, a contradiction. Therefore $\sigma(f)$ is not empty.

It now remains for us to show that $\sigma(f)$ is closed. From Theorem 9 we know the set of invertible elements of X is open.

Suppose $z \in \mathbb{C}$, $z \notin \sigma(f)$. Then $f - z$ is invertible so there exists a $\delta > 0$ such that g is invertible whenever $||f - z - g|| < \delta$. Assume $|w - z| < \delta$. Then $||f - z - (f - w)|| = ||w - z|| = |w - z| < \delta$, so $w \notin \sigma(f)$. Thus $\mathbb{C} - \sigma(f)$ is open and $\sigma(f)$ is closed. Now we know $\sigma(f)$ is a closed and bounded subset of $\mathbb{C}$ and is therefore compact.

<u>Theorem 12</u> (Gelfand-Mazur): If X is a Banach Algebra with identity in which each non-zero element has an inverse, then X is isomorphic to $\mathbb{C}$.

<u>Proof</u>: Let $f \in X$. Choose $z_1, z_2 \in \sigma(f)$, $z_1 \neq z_2$. At least one of the elements $f - z_1$ and $f - z_2$ must be invertible, since they cannot both be 0. Thus by Theorem 11, $\sigma(f)$ consists of a single element, say z_f, for each $f \in X$. Since $f - z_f$ is not invertible it must be 0, hence $f = z_f \cdot 1$ where 1 is the identity of X. The mapping $z_f \cdot 1 \to z_f$ is therefore an isomorphism of X onto $\mathbb{C}$.

<u>Theorem 13</u>: If X is an algebra then every proper ideal of X is contained in a maximal ideal. If, in addition, X is a Banach Algebra, then every maximal ideal of X is closed.

<u>Proof</u>: The first part of the theorem holds in any commutative ring with identity. Let I be a proper ideal of X. Let P be the collection of all proper ideals which contain I. P is partially ordered by set inclusion. Using the Hausdorff maximality principle, let M be the union of the ideals in some maximal linearly ordered subcollection Q of P. Then M is an ideal, $I \subset M$, $M \neq X$, since

no member of P contains the unit of X. Maximality of Q implies that M is a maximal ideal of X.

Now let M be a proper ideal in the Banach Algebra X. We wish to show that $\bar{M}$ is a proper ideal of X. We will first observe that $\bar{M}$ is a subspace of X. Let $f_1, f_2 \in \bar{M}$. Let $\delta > 0$. By definition of $\bar{M}$ there exists $g_1, g_2 \in M$ such that $||f_1 - g_1|| < \delta/2$ and $||f_2 - g_2|| < \delta/2$. Thus $||f_1 - g_1 + f_2 - g_2|| \leq ||f_1 - g_1|| + ||f_2 - g_2|| < \delta$ and $||(f_1 + f_2) - (g_1 + g_2)|| < \delta$. Since $g_1 + g_2 \in M$ we now know that $f_1 + f_2 \in \bar{M}$ and $\bar{M}$ is closed under addition. To show that $\bar{M}$ is closed under multiplication, let $f \in \bar{M}$, $z \in \mathbb{C}$. For $\delta > 0$ there exists $g \in M$ such that $||f - g|| < \delta/|z|$,

$$|z|\ ||f - g|| < \delta,$$

$$||zf - zg|| < \delta .$$

Since $zg \in M$, we know $zf \in \bar{M}$ for each $z \in \mathbb{C}$, $f \in \bar{M}$, and $\bar{M}$ is closed under scalar multiplication.

Now let $f \in X$, $g \in \bar{M}$. We wish to show that $fg \in \bar{M}$. We know that $g \in \bar{M}$ implies for each $\delta > 0$ there exists $g' \in M$ such that $||g - g'|| < \delta/||f||$,

$$||f||\ ||g - g'|| < \delta,$$

$$||fg - fg'|| < \delta .$$

Since $fg' \in M$ we know $fg \in \bar{M}$. Therefore $\bar{M}$ is an ideal of X.

Since M contains no invertible element of X and since the set of all invertible elements, V, is open, we know $M \cap V = \emptyset$, which implies $\bar{M} \cap V = \emptyset$. Therefore $\bar{M}$ is a proper ideal since V is nonempty.

Theorem 14: If ψ is a homomorphism from X to $\mathbb{C}$, then $X/\ker\psi$ is isomorphic to $\mathbb{C}$. Thus $\ker\psi$ is a maximal ideal.

Proof: Define ϕ: $X/\ker\psi \to \mathbb{C}$

$$f + \ker\psi \to \psi(f), \text{ for each } f \in X.$$

If $\phi(f + \ker\psi) = 0$ then $\psi(f) = 0$ and $f \in \ker\psi$. Therefore $f + \ker\psi = \ker\psi$ and ϕ is an isomorphism. Thus $X/\ker\psi$ is a field and by Theorem 1 $\ker\psi$ is a maximal ideal.

Lemma 1: Every maximal ideal of X is the kernel of some homomorphism from X to $\mathbb{C}$.

Proof: Let M be a maximal ideal of X. Then X/M is a field and, since M is closed, X/M is a Banach Algebra. By the Gelfand-Mazur Theorem (Theorem 12), there is an isomorphism π of X/M onto $\mathbb{C}$. Let $\phi = \pi \circ \psi$, where ψ is the homomorphism of X onto X/M with kernel M. Then ϕ is a homomorphism from X to $\mathbb{C}$ and $\ker\phi = M$.

Theorem 15: If X is a Banach Algebra, then there is a one-to-one correspondence between the set of all maximal ideals of X and the set of all homomorphisms from X to $\mathbb{C}$. Furthermore, each of these homomorphisms is continuous and has norm 1.

Proof: From Lemma 1 and Theorem 14 we know there exists a one-to-one correspondence between the set of all maximal ideals in X and the set of all homomorphisms from X to $\mathbb{C}$. Since each homomorphism has norm 1 by Theorem 6, we know each homomorphism is continuous by Theorem 4, and the theorem is proved.

CHAPTER III

THE SPACE OF MAXIMAL IDEALS

In this chapter we show the existence of a topology with respect to which the set of all homomorphisms of a Banach Algebra X is a compact Hausdorff space. Using this topology we then show that there is a norm-preserving homomorphism from X onto a subalgebra of continuous functions on a compact Hausdorff space, namely the space of all multiplicative linear functionals on X.

Definition 15: Consider the mapping $X \to X^{**} (X^{**} = (X^*)^*)$
$f \to \hat{f}$

defined by $\hat{f}(L) = L(f)$ for each $L \in X^*$ and $f \in X$. Thus we have $\hat{f} : X^* \to \mathbb{C}$ and $\hat{f}$ is evaluation at f.

$\hat{f} : L \to L(f)$

Remark 6: Observe that $|\hat{f}(L)| = |L(f)| \le ||L|| \; ||f|| = ||f|| \; ||L||$, which implies $||\hat{f}|| \le ||f||$. By the Hahn-Banach Theorem there exists $L_0 \in X^*$ such that $L_0(f) = ||f||$ and $||L_0|| = 1$[7]. Hence we have $|\hat{f}(L)| = |L(f)|$ for each $L \in X^*$, $|\hat{f}(L_0)| = |L_0(f)| = ||f|| = ||f|| \; ||L_0||$, and $||f|| \le ||\hat{f}||$. Therefore $||\hat{f}|| = ||f||$ for each $f \in X$ and the mapping $f \to \hat{f}$ is norm-preserving.

Definition 16: We call a basic neighborhood of H in X^* any set of the form $\{L \in X^* : |L(f_k) - H(f_k)| < \epsilon, \; k = 1,2,\ldots, n\}$, where $f_1, f_2, \ldots, f_n \in X$, $\epsilon > 0$. The topology for X^* for which the sets just described are a basis is the weakest topology for X^* with respect to which each $\hat{f}$ is continuous. This topology is called the weak-star topology for X^*.

For each $f \in X$ let $D_f = \{z \in \mathbb{C} : |z| \leq ||f||\}$. Let $P = \prod_{f \in X} D_f$. Then P is compact. An element of P is a function K defined on X such that $K(f) \in D_f$ for all $f \in X$. Let $S^* = \{L \in X^* : ||L|| \leq 1\}$. Then $S^* \subseteq P$. The product topology for P is the weakest topology such that $p_f : P \to D_f$ is continuous for all

$$\begin{aligned} p_f : P &\to D_f \\ K &\to K(f) \end{aligned}$$

$f \in X$. On S^* we require that $p_f : P \to D_f$ be continuous.

$$\begin{aligned} p_f : P &\to D_f \\ L &\to L(f) \\ L &\to \hat{f}(L) \end{aligned}$$

On S^*, $p_f = \hat{f}$. Thus on S^*, the product topology for P and the weak-star topology for X^* are the same.

Theorem 16: The set S^* is compact with respect to the weak-star topology.

Proof: We wish to show that S^* is closed in P. Let H be a limit point of S^* with respect to the product topology for P. Then for each $\epsilon > 0$, and $f \in X$, $\{K \in P : |p_f(K) - p_f(H)| < \epsilon\} = \{K \in P : |K(f) - H(f)| < \epsilon\}$ must contain an element of S^*, call it K_ϵ. Since $||K_\epsilon|| \leq 1$, $|K_\epsilon(f)| \leq ||f||$ and $|H(f)| - ||f|| \leq |H(f)| - |K_\epsilon(f)| \leq |H(f) - K_\epsilon(f)| < \epsilon$, $|H(f)| < \epsilon + ||f||$. This implies $|H(f)| \leq ||f||$ for each $f \in X$. Therefore $||H|| \leq 1$ and $H \in S^*$. Therefore S^* is closed in P and S^* is compact.

<u>Definition 17</u>: The set of all multiplicative linear functionals on X is denoted $M(X)$, and when given the weak-star topology (which it inherits as a subset of X^*), it is called the <u>space of maximal ideals in X</u>.

<u>Theorem 17</u>: The space of maximal ideals in X, $M(X)$, is a weak-star compact subset of S^*.

<u>Proof</u>: Clearly $M(X) \subseteq S^*$. Suppose $K \in S^*$ is a weak-star limit point of $M(X)$. Let $\epsilon > 0$, $f,g \in X$. Let $h_1 = f$, $h_2 = g$, $h_3 = fg$. Let $\delta = \min\left(\frac{\epsilon}{3}, \frac{\epsilon}{3||g||+1}, \frac{\epsilon}{3|K(f)|+1}\right)$. The set $N = \{L \in S^* : |K(h_i) - L(h_i)| < \delta,\ i = 1, 2, 3\}$ is a weak-star neighborhood of K. Therefore, there exists a $\phi \in M(X)$ such that $\phi \in N$. Thus $|K(f) - \phi(f)| < \delta$, $|K(g) - \phi(g)| < \delta$, $|K(fg) - \phi(fg)| < \delta$, and

$$|K(f)\,K(g) - K(f)\,\phi(g)| < |K(f)|\delta,$$
$$|\phi(g)\,K(f) - \phi(f)\,\phi(g)| < |\phi(g)|\delta,$$
$$|K(fg) - K(f)\,\phi(g)| < \delta + |\phi(g)|\delta,$$

so $|K(fg) - K(f)\,K(g)| < \delta + |\phi(g)|\,\delta + |K(f)|\,\delta < \epsilon$, since $|\phi(g)| \leq ||g||$. Thus K is a multiplicative linear functional and $K \in M(X)$.

Note: From this point on by $\hat{f}$ we mean $\hat{f}$ restricted to $M(X)$.

<u>Theorem 18</u>: The space of maximal ideals in X, $M(X)$, is weak-star Hausdorff.

<u>Proof</u>: Let $L_1, L_2 \in M(X)$. Choose $f \in X$ such that $L_1(f) \neq L_2(f)$ and let $|L_1(f) - L_2(f)| = \epsilon$. Then $K_1 = \{L \in M(X) : |L(f) - L_1(f)| < \epsilon/2\}$ is an open set about L_1 and $K_2 = \{L \in M(X) : |L(f) - L_2(f)| < \epsilon/2\}$ is an open set about L_2 and $K_1 \cap K_2 = \emptyset$.

<u>Definition 18</u>: If Y is a compact Hausdorff space, then $C(Y)$ denotes the set of all complex-valued continuous functions on Y.

<u>Remark 7</u>: The mapping $f \to \hat{f}$ maps X into the Banach Algebra $C(M(X))$. Observe that $C(M(X))$ is an algebra of continuous functions on a compact Hausdorff space.

<u>Theorem 19</u>: The mapping $f \to \hat{f}$ is a homomorphism.

<u>Proof</u>: Since $\widehat{f+g} : X^* \to \mathbb{C}$,

$$\widehat{f+g} : \phi \to \phi(f+g) = \phi(f) + \phi(g),$$

and $\hat{f} + \hat{g} : \phi \to \phi(f) + \phi(g)$ for each $\phi \in X^*$, then $\widehat{f+g} = \hat{f} + \hat{g}$. Since $\widehat{f \cdot g} : X^* \to \mathbb{C}$,

$$\widehat{f \cdot g} : \phi \to \phi(f \cdot g) = \phi(f) \cdot \phi(g),$$

and $\hat{f} \cdot \hat{g} : \phi \to \phi(f) \cdot \phi(g)$, for each $\phi \in X^*$, then $\widehat{f \cdot g} = \hat{f} \cdot \hat{g}$ and the proof is finished.

<u>Theorem 20</u>: The mapping $f \to \hat{f}$ maps X onto a closed subalgebra of $C(M(X))$.

<u>Proof</u>: By Theorem 19 we know that $\hat{X} = \{\hat{f} : f \in X\}$ is a subalgebra of $C(M(X))$. To prove that $\hat{X}$ is closed in $C(M(X))$, we show that it is a complete subspace of $C(M(X))$. Let $\{\hat{f}_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\hat{X}$. Then since X is a complete space and since $||\hat{f}_m - \hat{f}_n|| = ||f_m - f_n||$, $\{f_n\}_1^{\infty}$ has a limit g in X. Clearly $\hat{g}$ is the limit of $\{\hat{f}_n\}_1^{\infty}$ in X.

CHAPTER IV

THE SPACE OF MAXIMAL IDEALS FOR A AND FOR H^∞

This chapter is an investigation of the space of maximal ideals for two specific Banach Algebras. In one, everything about the space of maximal ideals is known. In the other, we find a much more complicated ideal structure.

Definition 19: Let A be the set of all complex-valued functions which are continuous on $\bar{U}$ (closed unit disc in $\mathbb{C}$) and analytic on U (open unit disc in $\mathbb{C}$) [4].

Remark 8: Each $f \in A$ is bounded since f is a continuous function on a compact set. Let $f \in A$ and let $||f|| = \text{lub}\,\{|f(z)| : z \in \bar{U}\}$. Suppose $\{f_n\}_1^\infty$ is a Cauchy sequence in A. Since $||f_m - f_n|| = \text{lub}\,\{|f_m(z) - f_n(z)| : z \in \bar{U}\}$, $\{f_n\}_1^\infty$ is uniformly convergent on $\bar{U}$. Thus $g(z) = \lim_{n\to\infty} f_n(z)$ is continuous on $\bar{U}$ and analytic on U and hence is in A. Therefore, with respect to $||\ ||$, A is a Banach Algebra.

Definition 20: Let H^∞ be the set of all bounded analytic functions on U [4]. Clearly $A \subseteq H$. For each $f \in H^\infty$, let $||f|| = \text{lub}\,\{|f(z)| : z \in U\}$. With respect to $||\ ||$, H^∞ is a Banach Algebra.

Remark 9: Let $z_0 \in \bar{U}$. Then $M_{z_0}(A) = \{f \in A : f(z_0) = 0\}$ is a maximal ideal in A since the mapping ϕ_{z_0} defined by

$\phi_{z_0} : A \to \mathbb{C}$ is a homomorphism with kernel M_{z_0}. The homomorphism

$\phi_{z_0}(f) = f(z_0)$

ϕ_{z_0} defined above is called an <u>evaluation</u> <u>map</u>. Similarly let $z \in U$. Then $M_z(H^\infty) = \{f \in H^\infty : f(z) = 0\}$ is a maximal ideal in H^∞.

Question: Are there any maximal ideals in A (or H^∞) which are not of the form M_z, for some $z \in \bar{U}$ (or U)?

We will first show that all maximal ideals in A are of the form M_z for some $z \in \bar{U}$. We will then show that the corresponding case does not hold in H^∞.

<u>Theorem 21</u>: If $f \in A$ and $\epsilon > 0$, then there exists a polynomial p such that $||f - p|| < \epsilon$, or, the polynomials are dense in A.

<u>Proof</u>: Recall that if h is analytic on a disc D with center 0, and $a_n = \dfrac{h^{(n)}(0)}{n!}$, $n = 0, 1, 2, \ldots$, then $h(z) = \sum_{n=0}^{\infty} a_n z^n$ for all z in D. Moreover, if D_0 is a compact subset of D and $\epsilon > 0$, there is a positive integer N such that $|f(z) - \sum_{n=0}^{m} a_n z^n| < \epsilon$ whenever $m \geq N$ and $z \in D_0$.

Let $f \in A$, and let $\epsilon > 0$. We are going to show that there is a polynomial p such that $||f - p|| < \epsilon$. Since f is continuous on $\bar{U}$, a compact set, there exists $\delta > 0$ such that $|z - w| < \delta$ implies $|f(z) - f(w)| < \epsilon$ for all $z, w \in U$. Let $0 < r < 1$. Define g_r by $g_r(z) = f(r \cdot z)$. Then $\text{Dom } g_r = \{z : |z| \leq 1/r\}$. Since $\bar{U}$ is a compact subset of $\text{Dom } g_r$, there exists a polynomial p_r such that $|g_r(z) - p_r(z)| < \epsilon/2$ for all $z \in \bar{U}$. Also there exists an r such that $|g_r(z) - f(z)| < \epsilon/2$ for all $z \in \bar{U}$. To see this, let δ be the positive number such that $|z - w| < \delta$ implies $|f(z) - f(w)| < \epsilon/2$, $z, w \in \bar{U}$. Let $1 - \delta < r < 1$. Then $|g_r(z) - f(z)| = |f(rz) - f(z)| < \epsilon/2$ since $|rz - z| < \delta$ for all $z \in \bar{U}$. Now we have $|f(z) - p_r(z)| \le |g_r(z) - p_r(z)| + |g_r(z) - f(z)| < \epsilon$.

Theorem 22: If ϕ is a multiplicative linear functional on A, and p is a polynomial in A, then $\phi(p) = p(\phi(I))$ where I is the identity on $\mathbb{C}$.

Proof: Suppose $p = a_n I^n + a_{n-1} I^{n-1} + \cdot \cdot \cdot + a_0$. Then
$\phi(p) = \phi(a_n I^n + a_{n-1} I^{n-1} + \cdot \cdot \cdot + a_0) =$
$a_n \phi(I)^n + a_{n-1} \phi(I)^{n-1} + \cdot \cdot \cdot + a_0 = p(\phi(I))$.

Theorem 23: If J is a maximal ideal in A, then there exists $z \in \bar{U}$ such that $J = M_z$. Since the multiplicative linear functional ϕ_z having M_z as its kernel is an evaluation map, this is equivalent to saying that every multiplicative linear functional on A is an evaluation.

Proof: We have just shown in Theorem 22 that $\phi(p) = p(\lambda)$, where $\lambda = \phi(I)$, for all polynomials p. It remains to be shown that $\phi(f) = f(\lambda)$ for all $f \in A$. Let $f \in A$. Since the polynomials are dense in A, there exists a sequence of polynomials in A, $\{p_n\}_{n=1}^{\infty}$, such that $\lim_{n\to\infty} p_n = f$. But ϕ is continuous so we have $\phi(f) = \lim_{n\to\infty} \phi(p_n) = \lim_{n\to\infty} p_n(\lambda) = f(\lambda)$.

Theorem 24: There is a one-to-one correspondence between the maximal ideals in A and the points in $\bar{U}$.

Proof: We know that for each $z \in \bar{U}$ there exists a maximal ideal M_z and for each maximal ideal M there exists $z \in \bar{U}$ such that $M = M_z$. Clearly if $M_z \neq M_w$ then $z \neq w$. Suppose $z_0, w_0 \in \bar{U}$ and $z_0 \neq w_0$. Let $f(z) = z - z_0$. Then $f \in M_{z_0}$, $f \notin M_{w_0}$. Therefore $M_{z_0} \neq M_{w_0}$ and the correspondence is one-to-one.

If we try to obtain a similar result for H^∞, we learn that H^∞ is a much more complicated Banach Algebra than A with regard to maximal ideals. To see this let $M = \{f \in H^\infty: \lim_{n\to\infty} f(\frac{n}{n+1}) = 0\}$. Then M is an ideal in H^∞, but for each $z_0 \in U$ the function $f(z) = 1 - z$ is in M and is not in $\ker \phi_{z_0}$, where ϕ_{z_0} is an evaluation map on H^∞. Thus $M \neq \ker \phi_z$ for all $z \in U$.

Consideration of convergent sequences in U in relation to the maximal ideals in A and H^∞ will again point out dissimilarities in the two spaces. Let $|z| = 1$. Let $\{w_p\}_1^\infty$ be a sequence in U such that $\lim_{p\to\infty} w_p = z$. Let $M_{\{w_p\}}(A) = \{f \in A: \lim_{p\to\infty} f(w_p) = 0\}$. Then $M_{\{w_p\}}(A)$ is an ideal in A and $M_{\{w_p\}}(A)$ depends only on the z to which $\{w_p\}_1^\infty$ converges, i.e., $M_{\{w_p\}} = M_z$. Thus from any two different sequences which converge to the same point on the boundary of U, we obtain the same ideal in A.

Question: If two different sequences converge to the same point on the boundary of U can we obtain two different ideals in H^∞?

To answer this question let $M_{\{w_p\}}(H^\infty) = \{f \in H^\infty: \lim_{p\to\infty} f(w_p) = 0\}$. Then $M_{\{w_p\}}(H^\infty)$ is an ideal in H^∞.

Define h by $h(z) = \frac{z+1}{z-1}$. If we let $z = x + iy$, then $\operatorname{Re} h(z) = \operatorname{Re}\left(\frac{z+1}{z-1}\right) = \operatorname{Re}\left(\frac{x+iy+1}{x+iy-1}\right) = \frac{x^2+y^2-1}{(x-1)^2+y^2} < 0$ for $z \varepsilon U$ since $|z| < 1$. Thus h maps U onto $\{z \varepsilon \mathbf{C} \mid \operatorname{Re} z < 0\}$. Also we can observe that $h = h^{-1}$. Now define the function G on U by $G(z) = \exp\left(\frac{z+1}{z-1}\right)$. Since $|G(z)| = \left|\exp\left(\frac{z+1}{z-1}\right)\right| = \exp\left(\operatorname{Re} \frac{z+1}{z-1}\right) < e^0 = 1$ from above, we have that G is bounded. Clearly G is analytic on U. Therefore $G \varepsilon H^\infty$. Since G is not continuous at $z = 1$, $G \notin A$. Let $s_n = \frac{n}{n+1}$. Then $\lim_{n\to\infty} s_n = 1$ and $\lim_{n\to\infty} G(s_n) =$

$$\lim_{n\to\infty} \exp\left(\frac{\frac{n}{n+1}+1}{\frac{n}{n+1}-1}\right) = \lim_{n\to\infty} \exp(-2n-1) = 0.$$ Thus $G \varepsilon M_{\{s_n\}}$.

Now let $r \varepsilon \mathbb{R}$ such that $r < 0$. Let $r_n = r + 2\pi ni$. Let $t_n = h(r_n)$. Then $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{r+2\pi ni+1}{r+2\pi ni-1} = 1$ and $\lim_{n\to\infty} G(t_n) =$

$$\lim_{n\to\infty} \exp\left(\frac{\frac{r+2\pi ni+1}{r+2\pi ni-1}+1}{\frac{r+2\pi ni+1}{r+2\pi ni-1}-1}\right) = \lim_{n\to\infty} e^{r+2\pi ni} = e^r.$$ Therefore $G \notin M_{\{t_n\}}$ and $M_{\{s_n\}} \neq M_{\{t_n\}}$. Thus we have exhibited two different sequences in U, converging to 1, from which we obtain two different ideals in H^∞.

Since every point in U other than 0 is e^w for some $w \varepsilon \mathbf{C}$ with $\operatorname{Re} w < 0$, we can now state the following:

<u>Theorem 25</u>: If $w \varepsilon \mathbf{C}$ and $0 < |w| < 1$, then there is a sequence $\{t_n\}_1^\infty$ in U such that $\lim_{n\to\infty} t_n = 1$ and $\lim_{n\to\infty} G(t_n) = w$.

<u>Proof</u>: Let $w \varepsilon \mathbf{C}$, $0 < |w| < 1$. Let $e^v = w$. Let $r_n = v + 2\pi ni$ and let $t_n = h(r_n)$ as defined above. Then $\lim_{n\to\infty} G(t_n) =$

$$\lim_{n\to\infty} \exp\left(\frac{\frac{v+2\pi ni+1}{v+2\pi ni-1}+1}{\frac{v+2\pi ni+1}{v+2\pi ni-1}-1}\right) = e^v = w$$ and the theorem is proved.

Definition 21: Let I be the identity function on $\mathbb{C}$. Then for each $\phi_z \in \mathcal{M}(A)$, $\phi_z(I) = I(z) \in \bar{U}$. Define a mapping $\rho : \mathcal{M}(A) \to \bar{U}$ by $\rho(\phi) = \phi(I)$. Observe $\rho(\phi) \in \bar{U}$ since $|\rho(\phi)| = |\phi(I)| \leq ||\phi||\ ||I|| = 1$.

Theorem 26: The set of maximal ideals in A, $\mathcal{M}(A)$, is homeomorphic to $\bar{U}$.

Proof: Clearly the above-defined ρ is a one-to-one and onto function. Since $\rho = \hat{I}$, ρ is continuous. Also ρ^{-1} is continuous since ρ is a one-to-one, onto, continuous mapping on a compact space.

Definition 22: Let I be the identity function on $\mathbb{C}$. Define a mapping $\pi: \mathcal{M}(H^\infty) \to \bar{U}$ by $\pi(\phi) = \phi(I)$. Clearly $|\pi(\phi)| = |\phi(I)| \leq 1$. To see that $|\pi(\phi)| = 1$ for some $\phi \in \mathcal{M}(H^\infty)$, we prove the following:

Lemma 2: If $z \in U$ and $\pi(\phi) = z$, then $\phi = \phi_z$.

Proof: Let $z_0 \in U$. Suppose $\phi \in \mathcal{M}(H^\infty)$ and $\pi(\phi) = z_0$. Suppose $f \in M_{z_0}$. We wish to show that $\phi(f) = 0$. Since each function in H^∞ is an analytic function on U, f has a power series representation, $f(z) = \sum_{p=0}^{\infty} \frac{f^{(p)}(z_0)}{p!}(z - z_0)^p =$

$$f(z_0) + \sum_{p=1}^{\infty} \frac{f^{(p)}(z_0)}{p!}(z - z_0)^p = \sum_{p=1}^{\infty} \frac{f^{(p)}(z_0)}{p!}(z - z_0)^p, \text{ since}$$

$f(z_0) = 0$. Now $(z - z_0)$ is a factor of $f(z)$ and $f(z) = (z - z_0)\, g(z)$, or $f = (I - z_0)g$. Thus $\phi(f) = \phi((I - z_0)g) = (\phi(I) - z_0)\, \phi(g) = (z_0 - z_0)\, \phi(g) = 0$. Now we know $M_{z_0} \subseteq \ker \phi$.

Let $j \in H^\infty$. Let $h(z) = j(z) - j(z_0)$. Then $h \in M_{z_0}$, so $\phi(h) = 0$. Now $j(z) = h(z) + j(z_0)$ and $j = j(z_0) + h$. Thus $\phi(j) = j(z_0) + \phi(h) = j(z_0) = \phi_{z_0}(j)$. Therefore $\phi = \phi_{z_0}$.

Remark 10: We can observe that $\pi^{-1}(U)$ is open in $M(H^\infty)$.

Theorem 27: Let $\Delta = \pi^{-1}(U)$. Then $\pi^{-1}\big|_U : U \to \Delta$ is a homeomorphism.

Proof: Since $\pi = \hat{I}$, π is continuous and $\pi\big|_\Delta$ is continuous. By Lemma 2, $\pi\big|_\Delta$ is one-to-one. Clearly $\pi^{-1}\big|_U$ is one-to-one. It remains to be shown that $\pi^{-1}\big|_U$ is continuous. To see this let $\phi_{z_0} \in \Delta$. Let N be a basic open set containing ϕ_{z_0}. Then $N = \{\phi_z : |\phi_z(f_i) - \phi_{z_0}(f_i)| < \epsilon,\ i = 1, . . ., n\}$ for some $\epsilon > 0$ and $f_1, . . ., f_n \in H^\infty$. We wish to show that there is a neighborhood N' of $z_0 = \pi(\phi_{z_0})$ such that $\pi^{-1}(N') \subseteq N$. Since each f_i, $i = 1, . . ., n$, is continuous, we know that for each f_i there exists a $\delta_i > 0$ such that $|f_i(z) - f_i(z_0)| < \epsilon$ whenever $|z - z_0| < \delta_i$. Let $\delta = \min(\delta_1, . . ., \delta_n)$. Choose a δ-neighborhood about z_0, call it N_δ. Then $N_\delta = \{\phi_{z'}(I) : |\phi_{z'}(I) - \phi_{z_0}(I)| < \delta\} = \{z : |z - z_0| < \delta\}$. Now we have $|\phi_{z'}(f_i) - \phi_{z_0}(f_i)| < \epsilon$ for $i = 1, . . ., n$. Hence $\pi^{-1}(\phi_{z'}(I)) = \phi_{z'} \in N$ and $\pi^{-1}(N_\delta) \subseteq N$.

Definition 23: If $\alpha \in \mathbf{C}$ and $|\alpha| = 1$, let $\pi^{-1}(\alpha) = M_\alpha$. Observe that M_α is a closed subset of $M(H^\infty)$.

Theorem 28: Suppose $f \in H^\infty$ and $|\alpha| = 1$. Let $\{s_n\}$ be a sequence in U with limit α, and suppose f has a limit on s, i.e., $w = \lim_{n\to\infty} f(s_n)$ exists. Then there exists a $\phi \in \mathcal{M}_\alpha$ such that $\phi(f) = w$.

Proof: Let $M_{\{s_n\}} = \{g \in H^\infty : g(s_n) \to 0 \text{ as } n \to \infty\}$. Then $M_{\{s_n\}}$ is an ideal in H^∞. Let $g(z) = f(z) - w$. Then $g \in M_{\{s_n\}}$. Let $h(z) = z - \alpha$, i.e., $h = I - \alpha$. Then $h \in M_{\{s_n\}}$. Now $M_{\{s_n\}}$ is contained in some maximal ideal M in H^∞ and M is the kernel of some homomorphism ϕ from H^∞ to $\mathbb{C}$. Since $g,h \in M_{\{s_n\}}$, then $g,h \in \ker \phi$ and $\phi(g) = 0$, $\phi(h) = 0$. Now

$$\begin{aligned} \phi(h) = \phi(I - \alpha) &= 0, \\ \phi(I) - \alpha &= 0, \\ \phi(I) &= \alpha, \end{aligned}$$

and thus $\phi \in \mathcal{M}_\alpha$. Since

$$\begin{aligned} \phi(g) = \phi(f - w) = 0, \quad \phi(f) - w &= 0, \\ \phi(f) &= w, \end{aligned}$$

and the theorem is proved.

Theorem 29: Let $f \in H^\infty$. Then $\hat{f}$ is constant on $\mathcal{M}_\alpha$ if and only if f has a continuous extension to $U \cup \{\alpha\}$.

Proof: Suppose f is constant on $\mathcal{M}_\alpha$, and let w be the constant value of f. Suppose there exists a sequence $\{s_n\}_1^\infty$ in U with limit α such that $\lim_{n\to\infty} f(s_n) \neq w$. Then there exists an $\epsilon > 0$ and a subsequence $\{t_n\}_1^\infty$ of $\{s_n\}_1^\infty$ such that $|f(t_n) - w| \geq \epsilon$ for all n. The sequence $\{f(t_n)\}_1^\infty$ is bounded since $f \in H^\infty$. Hence it has a convergent subsequence $\{f(v_n)\}_1^\infty$. The sequence $\{v_n\}_1^\infty$ has limit α, so from the previous theorem there exists $\phi \in \mathcal{M}_\alpha$ such that $\phi(f) = \lim_{n\to\infty} f(v_n) = w$. This is a contradiction to $|f(t_n) - w| \geq \epsilon$ for all n, so for every sequence $\{s_n\}_1^\infty$ in U with limit α, we have $\lim_{n\to\infty} f(s_n) = w$. If we define $f(\alpha) = w$, we have f continuous on $U \cup \{\alpha\}$.

Now suppose f has a continuous extension to $U \cup \{\alpha\}$. Let $f(\alpha) = w$. Let $h(z) = f(z) - w$ and $g(z) = 1/2(1 + \alpha^* z)$, where $z \in U$ and α^* is the conjugate of α. Observe that $|g(z)| < 1$ for each z in U. We wish to show that for each $\phi \in M_\alpha$, $\phi(h) = 0$. If $\phi \in M_\alpha$, then $\phi(g) = 1$. We need to show that $\lim_{n\to\infty} g^n h = 0$. Let $\epsilon > 0$. Since h is continuous at α, there exists a $\delta > 0$ such that if $z \in U$ and $|z - \alpha| < \delta$, then $|h(z)| < \epsilon$. If $z \in U$ and $|z - \alpha| < \delta$, then $|g(z)|^n \, |h(z)| < \epsilon$. Let $r = \operatorname{lub}_{\substack{z\in U \\ |z-\alpha|\geq\delta}} |g(z)|$.

Then $r < 1$, since $|g(z)| = 1$ only at α. There exists a positive integer N such that if $n \geq N$, then $r^n < \frac{\epsilon}{||h||}$. Thus if $z \in U$ and $|z - \alpha| \geq \delta$, then $|g(z)|^n \, |h(z)| \leq r^n \, ||h|| < \epsilon$. Therefore $||g^n h|| < \epsilon$, since $||g^n h|| = \operatorname{lub}_{|z|<1} |g(z)^n h(z)| = \operatorname{lub}_{|z|<1} |g(z)|^n \, |h(z)|$.

We now have that $||g^n h|| \to 0$ as $n \to \infty$. For each $\phi \in M_\alpha$, $\phi(g^n h) = \phi(g^n)\,\phi(h) = \phi(g)^n\,\phi(h) = \phi(h)$. Yet as $n \to \infty$, $\phi(g^n h) \to 0$, so $\phi(h) = 0$. Now $\hat{f}(\phi) = \phi(f) = \phi(h) + w = w$ for each $\phi \in M_\alpha$. Thus ϕ is constant on M_α, which completes the proof.

Corollary 1: Let $G(z) = \exp(\frac{z+1}{z-1})$. Then $\hat{G}$ is not constant on M_1.

Proof: The proof follows immediately from the fact that $G(z)$ is not continuous on $U \cup \{1\}$.

Corollary 2: If $|\alpha| = 1$, then there exists $f \in H^\infty$ such that $\hat{f}$ is not constant on M_α.

Proof: Let $f(z) = G(1/\alpha\, z)$. Then f has a continuous extension to $U \cup \{\alpha\}$ if and only if G has a continuous extension to $U \cup \{1\}$.

Remark 11: Observe that we now know π is not a one-to-one mapping.

Theorem 30: If $w \in U$ and $w \neq 0$, then there exists $\phi \in M_1$ such that $\phi(G) = w$.

Proof: Let $w \in U$, $w \neq 0$, and let $k = |w|$. Then there is exactly one number r such that $k = e^r$ and there is exactly one number $v \in [0, 2\pi)$ such that $w = ke^{iv}$. Define the sequence $\{s_n\}_0^\infty$ by $s_n = \frac{r+vi+2\pi ni+1}{r+vi+2\pi ni-1}$. Then $\lim_{n\to\infty} s_n = 1$ and $\lim_{n\to\infty} G(s_n) = e^r e^{iv} = ke^{iv} = w$. Therefore by Theorem 28, there exists $\phi \in M_1$ such that $\phi(G) = w$.

Definition 24: A boundary for a subalgebra X of $C(Y)$, Y a compact Hausdorff space, is a closed subset S of Y such that if $f \in X$, then $\max_{t\in Y} |f(t)| = \max_{t \in S} |f(t)|$. The smallest boundary for X is called the Silov boundary [5].

Theorem 31: The Silov boundary for $\hat{A}$ is $S = \{\phi_z : |z| = 1\}$. (We sometimes say the Silov boundary for A is $\overline{U} - U$).

Proof: We know that S is closed since it is the inverse image of a closed set $\{z : |z| = 1\}$ under the continuous mapping ρ defined in Definition 21. Since $\max_{\phi_z \in M(A)} |\hat{f}(\phi_z)| = \max_{z\in\overline{U}} |f(z)|$, we know that the Silov boundary of A is a subset of S by the maximum modulus principle [7]. For each $\alpha \in \bar{U}$, $|\alpha| = 1$, we can build a function $g_\alpha \in A$ where $g_\alpha(z) = 1/2(1 + \alpha^* z)$. Then g_α takes on its maximum absolute value at α. Thus S is the Silov boundary for $\hat{A}$.

<u>Theorem 32</u>: If ϕ is in the Silov boundary of $\hat{H}^\infty$, then $\phi \notin \Delta$, i.e., $\phi \neq \phi_z$ for each $z \in U$.

<u>Proof</u>: The Silov boundary of $\hat{H}^\infty$ is a closed subset S of $\hat{H}^\infty$ such that $\max\limits_{\phi \in M(H^\infty)} |\hat{f}(\phi)| = \max\limits_{\phi \in S} |\hat{f}(\phi)|$ for each $\hat{f} \in \hat{H}^\infty$.

Assume $z \in U$ and ϕ_z is in the Silov boundary of $\hat{H}^\infty$. Then there exists $\hat{f} \in \hat{H}^\infty$ such that $|\hat{f}(\phi_z)| \geq |\hat{f}(\phi)|$ for each $\phi \in M(H^\infty)$. But then $|\hat{f}(\phi_z)| \geq |\hat{f}(\phi_w)|$ for each $w \in U$, and $|f(z)| \geq |f(w)|$ for each $w \in U$. This contradicts the Maximum Principle. Therefore ϕ_z is not in the Silov boundary of $\hat{H}^\infty$.

<u>Theorem 33</u>: If ϕ is in the Silov boundary of $\hat{H}^\infty$, then $|\hat{G}(\phi)| = 1$.

<u>Proof</u>: We know $|\hat{G}(\phi)| \leq ||\hat{G}||\ ||\phi|| = ||G||\ ||\phi|| = 1$. Assume ϕ is in the Silov boundary and $|\hat{G}(\phi)| < 1$. Then there exists a neighborhood N of ϕ such that $|\hat{G}(\psi)| < 1$ for each $\psi \in N$ since $\hat{G}$ is continuous. There exists $f \in H^\infty$ such that $\hat{f}$ does not take on its maximum absolute value on the complement of N. Otherwise, the complement of N would contain the Silov boundary. We have $||fG|| = \max\limits_{\phi \in M(H^\infty)} |f(\phi)\ G(\phi)| < ||f||\ ||G||$.

Since $\hat{\hat{f}}\hat{G} = \hat{f}\hat{G}$, we have $||\hat{f}\hat{G}|| < ||\hat{f}||$, Define a sequence $\{s_n\}_1^\infty$

$$||\widehat{fG}|| < ||\hat{f}||,$$

$$||fG|| < ||f||.$$

by $s_n = -1/n + ni$, and a sequence $\{t_n\}_1^\infty$ by $t_n = \frac{-1/n+ni+1}{-1/n+ni-1}$.
Then $\lim_{n\to\infty} t_n = 1$ and $\lim_{n\to\infty} |G(t_n)| = \lim_{n\to\infty} |\exp(s_n)| = \lim_{n\to\infty} e^{-1/n} = 1$.
There exists a sequence $\{r_n\}_1^\infty$ in U such that $|r_n| > |t_n|$ and $\lim_{n\to\infty} |f(r_n)| = ||f||$, and there is a convergent subsequence $\{v_n\}_1^\infty$ of $\{r_n\}_1^\infty$. Let $\alpha = \lim_{n\to\infty} v_n$. Then $|\alpha| = 1$. Assume $\alpha \neq 1$. Then $\lim_{n\to\infty} |G(v_n)| = 1$, so $\lim_{n\to\infty} |f(v_n)\, G(v_n)| = \lim_{n\to\infty} |f(v_n)|\, |G(v_n)| = ||f||$. Therefore $||f|| \leq ||fG||$. This contradiction shows that $\alpha = 1$. Now for $\alpha = 1$ we again have $\lim_{n\to\infty} |G(v_n)| = 1$, and $\lim_{n\to\infty} |f(v_n)\, G(v_n)| = ||f||$ and we reach the same contradiction as above. Therefore $|G(\phi)| = 1$.

Theorem 34: The Silov boundary of $\hat{H}^\infty$ is a proper subset of $M(H^\infty) - \Delta$.

Proof: Let $s_n = \frac{n}{n+1}$. Then $\lim_{n\to\infty} G(s_n) = 0$, $\lim_{n\to\infty} \phi_{s_n}(G) = 0$, $\lim_{n\to\infty} \hat{G}(\phi_{s_n}) = 0$. Consider the sequence $\{\phi_{s_n}\}_1^\infty$. Since each $\phi_{s_n} \in M(H^\infty)$ and since any sequence on a compact space has a cluster point, $\{\phi_{s_n}\}_1^\infty$ has a cluster point, call it ψ. Let $\epsilon > 0$. Let $N = \{\phi \in M(H^\infty) : |\hat{G}(\phi) - \hat{G}(\psi)| < \epsilon\}$. Then N is a neighborhood of ψ since $\hat{G}$ is continuous. There exists a positive integer n such that $\phi_{s_n} \in N$ and $|G(\phi_{s_n})| < \epsilon$. We have $|\hat{G}(\psi)| - |\hat{G}(\phi_{s_n})| \leq |\hat{G}(\psi) - \hat{G}(\phi_{s_n})| < \epsilon$ and $|\hat{G}(\psi)| < 2\epsilon$. Thus $\hat{G}(\psi) = 0$ and $\psi \notin \Delta$ since $\exp z \neq 0$ for all $z \in U$. But $\psi \notin$ Silov boundary of $\hat{H}^\infty$ by the previous theorem. Hence we have an element of $M(H^\infty) - \Delta$ which is not in the Silov boundary of H^∞ and the theorem is proved.

SUMMARY

In this paper we have shown that the ideal structure of H^∞ is exceedingly rich. This observation leads the way to a wealth of questions concerning H^∞ and $M(H^\infty)$. We will mention a few. Is each M_α connected? Is $M(H^\infty) - \Delta$ connected? The answer to each of these two questions is "yes", as is shown in a paper by Kenneth Hoffman, "A Note on the Paper of I. J. Schark," *Journal of Mathematics and Mechanics*, X (1961). One might try, to characterize the set of all isometries, i.e., distance-preserving linear transformations, of H^∞. This has been done by M. Nagasawa in "Isomorphisms between Commutative Banach Algebras, with Application to Rings of Analytic Functions," *Kokai Mathematics Seminar Report*, XI (1959). Apparently, the following question remains unanswered: Is Δ dense in $M(H^\infty)$?

A reference for much of the information in Chapter IV of this thesis is [8], a paper by I. J. Schark, who first developed this material.

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